1. *State true or false. Justify your answers.*

(i) Suppose that A and B are nonsingular $n \times n$ matrices. Then $A + B$ is nonsingular matrix.

Solution: False. If A is nonsingular then so is $-A$, bu then $A - A = 0$ is a singular matrix. $\square$

(ii) Let A be an $n \times n$ matrix such that the sum of elements in each row of A is zero. Then A is a singular matrix.

Solution: True. Since the sum of elements in each row of A is zero, one of the columns (say n^{th}-column) can be written as linear combination of other columns. Thus, it is clear that A is a singular matrix.
Another way to prove this fact is as follows: Let A is nonsingular, also let us consider the vector $v = (1, \cdots, 1) \in \mathbb{K}^n$. Then we have $A.v = 0$, which is in contradiction (since given that A is invertible, $Ax = 0$ has a unique solution for $x \in \mathbb{K}^n$). $\square$

(iii) Let $C[-\pi, \pi]$ be the vector space of all continuous functions defined on the interval $[-\pi, \pi]$. The subset $\{Cos(x), Sin(x)\}$ in $C[-\pi, \pi]$ is linearly independent.

Solution: True. Let $a_1, a_2 \in \mathbb{R}$ such that

$$a_1.Cos(x) + a_2.Sin(x) = 0.$$

Let $x = \pi/2$, then $a_2 = 0$.

Similarly, if $x = 0$, then $a_1 = 0$. Hence, $\{Cos(x), Sin(x)\}$ is linearly independent subset in $C[-\pi, \pi]$. $\square$

(iv) Let A be an $n \times n$ matrix. If $rank(A) \neq n$, then 0 is an eigenvalue of A.

Solution: True. Let us note that $rank(A) \neq n$ if and only if $det(A) = 0$ and $det(A)$ is the product of eigen values of A. Therefore, if $det(A) = 0$, then 0 is an eigen value of A. $\square$

(v) Let $A,\ B$ be $n \times n$ matrices with A invertible. Then nullity(AB) = nullity(B).

Solution: True. Let us note that since A is invertible, for any $x \in \mathbf{K}^n$, we have

$$Ax = 0 \text{ if and only if } x = 0.$$

Thus, $ABx = 0$ if and only if $Bx = 0$, which, in turn, implies that

$$\text{nullspace}(AB) = \text{nullspace}(B).$$

Hence, nullity(AB) = nullity(B). $\square$

2. Let

$$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 1 & 2 & 5 & 4 \end{bmatrix}$$

(a) Find the solutions to the system of equations $AX = 0$.

Solution: Let us first change A into echelon form.

$R_2 \to R_2 - 3.R_1, R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -5 & -7 \\ 0 & 1 & 3 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -5 & -7 \\ 0 & 0 & 8 & 8 \end{bmatrix}$$

$R_3 \to R_3/8$

$$\sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -5 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Let

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}.$$

So, solutions of $AX = 0$ is given by the following expression.

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -5 & -7 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = 0$$

Thus, solution is given by

$$(x, y, z, w) = \{k(-3, 2, -1, 1) : k \in \mathbb{R}\}$$

□

(b) Find a basis for the solution space of $AX = 0$.

Solution: Dimension of solution space= number of unknowns-rank(A)=1. Thus, basis for the solution space of $AX = 0$ is {(-3,2,-1,1)}. □

3. Let P_3 denote the vector space of all polynomials with real coefficients of degree less than or equal to 3. Let $S = \{1 + x, 1 + x^2, x - x^2 + 2x^3, 1 - x - x^2\}$.

   (a) Show that S is a basis for P_3.

   Solution: Since number of elements in S is equal to the dimension of P_3, it is enough to show that S is linearly independent set.
   Let $a.(1 + x) + b.(1 + x^2) + c.(x - x^2 + 2x^3) + d.(1 - x - x^2) = 0$
   i.e., $(a + b + d).1 + (a + c - d).x + (b - c - d).x^2 + (2c).x^3 = 0$
   Writing the matrix form in terms of the standard basis $\{1, x, x^2, x^3\}$, we have

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

   Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

   Then using row transformations it follows that $rank(A) = 4$, i.e. there exists a unique solution given by $a = b = c = d = 0$. Hence, the set S is a basis for P_3. □

   (b) Find the change of basis matrix with respect to the standard basis $\{1, x, x^2, x^3\}$.

   Solution: Let us write the elements of the basis S in terms of the standard basis.

$$\begin{aligned} 1 + x &= 1.(1) + 1.(x) + 0.(x^2) + 0.(x^3) \\ 1 + x^2 &= 1.(1) + 0.(x) + 1(x^2) + 0.(x^3) \\ x - x^2 + 2x^3 &= 0.(1) + 1.(x) + (-1)(x^2) + 2.(x^3) \\ 1 - x - x^2 &= 1.(1) + (-1)(x) + (-1)(x^2) + 0.(x^3) \end{aligned}$$

   Therefore, the change of basis matrix is given by

$$P = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix}.$$

   □

   (c) Find the co-ordinate vector for the polynomial $f(x) = -3 + 2x^3$in terms of the basis S.

   Solution: Let us write $f(x)$ as a linear combination of the elements in S

$$f(x) = a.(1 + x) + b.(1 + x^2) + c.(x - x^2 + 2x^3) + d.(1 - x - x^2) = 0.$$

   i.e.,

$$-3 + 2x^3 = (a + b + d).1 + (a + c - d).x + (b - c - d).x^2 + (2c).x^3$$

On comparing both sides, we get following system of equations.

$$\begin{aligned} a+b+d &= -3 \\ a+c-d &= 0 \\ b-c-d &= 0 \\ 2c &= 2. \end{aligned}$$

The solution of the above system is given by

$$c = 1, d = -1, a = -2, b = 0.$$

Thus, the co-ordinate of $f(x)$ in terms of the basis S is given by

$$[f(x)]_S = \begin{bmatrix} -2 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

□

4. Let W_1 be the space of $n \times n$ matrices with trace zero. Find a subspace W_2 of $\mathbb{R}^{n\times n}$ such that $\mathbb{R}^{n\times n} = W_1 \oplus W_2$.

Solution: Note that $W_1 = \{A = (a_{ij})_{1\le i,j\le n} : \sum_{i=1}^{n} a_{ii} = 0\}$. Clearly, for a fixed $i \in \{1, 2, \ldots, n\}$, $a_{ii} = -\sum_{j\neq i} a_{jj}$.
i.e.,

$$dim(W_1) = n^2 - 1.$$

Let us define

$$W_2 = \{B = (b_{ij})_{1\le i,j\le n} : b_{ij} = 0 \forall (i,j) \neq (1,1)\}.$$

It is clear that $W_1 \cap W_2 = \{0\}$.
Now, let $C = (c_{ij}) \in \mathbb{R}^{n\times n}$. We define $A = (a_{ij}) \in W_1$, and $B = (b_{ij}) \in W_2$ such that $C = A + B$.

$$\begin{aligned} a_{ij} &= c_{ij} \qquad \forall i \neq j; \\ a_{ii} &= c_{ii} \qquad \forall i = 2, 3, \ldots, n; \\ a_{11} &= -\sum_{i=2}^{n} a_{ii}; \\ b_{11} &= -\sum_{i=1}^{n} a_{ii}; \\ b_{ij} &= 0 \qquad \forall (i,j) \neq (1,1). \end{aligned}$$

It immediately follows that $C = A + B$, i.e. $\mathbb{R}^{n\times n} \subset W_1 + W_2$. Therefore,

$$\mathbb{R}^{n\times n} = W_1 \oplus W_2.$$

□

5. (a) State and prove rank-nullity theorem for a linear transformation $T : V \to W$, where V is a finite dimensional vector space.

Solution: Let $dim(V) = n$, also let us fix a basis $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ of the vector space V. Let us assume that matrix of T with respect to the basis $\mathcal{B}$ is given by $A = [T]_{\mathcal{B}}$. If rank$(A) = n$, then the only solution to $AX = 0$ is the trivial solution $X = 0$. Hence, in this case, nullspace$(A) = 0$, so nullity$(A) = 0$ and the following equation holds.

$$\text{rank}(T) + \text{nullity}(T) = n.$$

Here, we used the fact that rank$(T) =$ rank(A), and nullity$(T) =$ nullity(A).

Now, we assume that rank$(A) = r < n$. In this case, there are $n-r > 0$ free variables in the solution to $AX = 0$. Let $t_1, t_2, \ldots, t_{n-r}$ denote these free variables (chosen as those variables not attached to a leading one in any row-echelon form of A), and let $X_1, X_2, \ldots, X_{n-r}$ denote the solutions obtained by sequentially setting each free variable to 1 and the remaining free variables to zero. Note that $\{X_1, X_2, \ldots, X_{n-r}\}$ is a linearly independent set. Every solution to $AX = 0$ is a linear combination of $X_1, X_2, \ldots, X_{n-r}$, i.e.,

$$X = t_1.X_1 + t_2.X_2 + \ldots + t_{n-r}.X_{n-r},$$

which shows that the set $\{X_1, X_2, \ldots, X_{n-r}\}$ spans nullspace(A). Thus, the set $\{X_1, X_2, \ldots, X_{n-r}\}$ is a basis for nullspace(A), and nullity$(A) = n - r$. Therefore, the following equation holds.

$$\text{rank}(T) + \text{nullity}(T) = n.$$

□

(b) Let $T : V \to V$ be a linear operator on a vector space of dimension 2. Assume that T is not a multiplication by a scalar. Prove that there is a vector $v \in V$ such that $\mathcal{B} = \{v, T(v)\}$ is a basis of V, and describe the matrix of T with respect to the basis $\mathcal{B}$.

Solution: Let us assume that for each $v \in V$, the set $\{v, T(v)\}$ is linearly dependent. i.e., there exist $a, b \in \mathbb{K}$ not simultaneously zero such that

$$a.v + b.T(v) = 0.$$

Case 1. If $a \neq 0, b = 0$, then $a.v = 0 \;\; \forall v \in V$ implies that $a = 0$, which is in contradiction.

Case 2. If $a = 0, b \neq 0$, then $b.T(v) = 0 \;\; \forall v \in V$ implies that $T(v) = 0 \;\; \forall v \in V$, i.e., T is a multiplication by a scalar, which is in contradiction.

Hence, $a, b \neq 0$, which in turn, implies that $T(v) = (a/b).v$ for all $v \in V$. Again, T is a multiplication by a scalar, which is in contradiction. Therefore, there exists a non-zero vector $v \in V$ such that $\{v, T(v)\}$ is a linearly independent set. So, $\mathcal{B}$ is a basis of V.

Next, $T(V) = 0.v + 1.T(v)$, and $T^2(v) = \alpha.v + \beta.T(v)$ for some $\alpha, \beta \in \mathbb{K}$. Thus, matrix of T is given as follows.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 0 & \alpha \\ 1 & \beta \end{bmatrix}.$$

□

6. Define the map $T : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_2 + x_3 \\ x_3 \end{bmatrix}.$$

(a) Find the matrix of T with respect to the standard basis.

Solution: Let us write the images of the elements in the standard basis $\mathcal{S}$ under the map T.

$$\begin{aligned} T(1,0,0)^t =& (1,0,0)^t \\ T(0,1,0)^t =& (1,1,0)^t \\ T(0,0,1)^t =& (1,1,1)^t. \end{aligned}$$

Then, matrix of T with respect to the standard basis $\mathcal{S}$ is given by

$$[T]_{\mathcal{S}} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(b) Find the matrix of T with respect to the basis $\mathcal{B} = \{(1,1,0)^t, (1,-1,0)^t, (-1,0,-1)^t\}$.

Solution: If $T(1,1,0^t) = a.(1,1,0)^t + b.(1,-1,0)^t + c.(-1,0,-1)^t$, then we get the following expression.

$$(2,1,0) = (a+b-c, a-b, -c).$$

Clearly, $c = 0$, $a = 3/2$, and $b = 1/2$. i.e.,

$$[T(1,1,0)^t]_{\mathcal{B}} = \begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix}.$$

Similar calculation yields the following co-ordinates.

$$[T(1,-1,0)^t]_{\mathcal{B}} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}, \text{ and } [T(-1,0,-1)^t]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore, the matrix of T with respect to $\mathcal{B}$ is given as follows.

$$[T]_{\mathcal{B}} = \begin{bmatrix} 3/2 & -1/2 & -1 \\ 1/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

□

(c) Describe the null space and the range of T. Also find the rank and the nullity of T.

Solution: Null space of T is given by

$$N(T) = \{X \in \mathbb{R}^3 : T(X) = 0\}.$$

i.e., for $X = (x_1, x_2, x_3)^t \in N(T)$, we have

$$T(x_1, x_2, x_3)^t = (x_1 + x_2 + x_3, x_2 + x_3, x_3)^t = (0, 0, 0)^t,$$

which implies that $x_1 = x_2 = x_3 = 0$. Hence, $N(T) = \{0\}$.

If range of T is denoted by $R(T)$, then by Rank-Nullity theorem,

$$\dim(R(T)) = 3 = \text{ dimension of codomain}.1$$

i.e., $R(T) = \mathbb{R}^3$. Moreover, rank of $T = 3$ and Nullity of $T = 0$. □